Multi–Instantons, Supersymmetry and Topological Field Theories

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ABSTRACT

In this letter we argue that instanton–dominated Green's functions in $N = 2$ Super Yang–Mills theories can be equivalently computed either using the so–called constrained instanton method or making reference to the topological twisted version of the theory. Defining an appropriate BRST operator (as a supersymmetry plus a gauge variation), we also show that the expansion coefficients of the Seiberg–Witten effective action for the low–energy degrees of freedom can be written as integrals of total derivatives over the moduli space of self–dual gauge connections.

1 Introduction

Much progress has been made in recent years in understanding non–perturbative phenomena in globally supersymmetric (SUSY) gauge theories. In a seminal paper [1], Seiberg and Witten calculated all the non–perturbative contributions to the holomorphic effective action for a Super Yang–Mills (SYM) theory with $N = 2$ global supersymmetry using certain ansätze dictated by physical intuitions. To check the correctness of this result, the contributions to the prepotential $\mathcal{F}$ coming from instantons of winding number one and two were computed directly in the Coulomb phase of the theory, by using a saddle point approximation for the calculation of the relevant correlators [2, 3, 4]. The results were found to be in agreement with the expression of $\mathcal{F}$ derived in [1].

These successful checks raise a number of questions which are the motivations of our investigation. The most compelling one is probably the following: how comes that a saddle point approximation, in which only quadratic terms are retained in the expansion of the action, is able to give the correct result? In order to answer this question and extend previous computations, we are led to consider the instanton calculus in the framework of the topological twisted version of $N = 2$ SYM, the so–called Topological Yang–Mills theory (TYM) [5]. As we will show later, the "traditional" instanton calculus (the semiclassical expansion) and computations carried on in the TYM framework give the same result. We will extend the standard framework of TYM to encompass the case in which the scalar field acquires a vacuum expectation value (v.e.v.). To this end, we define a BRST operator for the "extended" TYM and find how it acts on the moduli space of (anti–)self–dual gauge connections, $\mathcal{M}^+$ ($\mathcal{M}^-$), realized via the ADHM construction [6].

From a geometrical point of view, this BRST operator is on $\mathcal{M}^+$ the exterior derivative. We exploit this fact to show that all correlators made of insertions of gauge invariant polynomials in the scalar ϕ, both with vanishing and non–vanishing v.e.v., can be written as total derivatives on $\mathcal{M}^+$; an example is given by the modulus $u =< \mathrm{Tr}\phi^2 >$, which parametrizes the space of quantum vacua of the $N = 2$ SYM theory with gauge group $SU(2)$. In turn this result means that the only relevant contributions to this class of correlators come from $\partial\mathcal{M}^+$. To have an idea of what the behavior of boundary terms is, we can look at the compactified moduli space $\overline{\mathcal{M}}_k$. In [7] it was shown that $\partial\overline{\mathcal{M}}_k$ can be decomposed into the union of lower moduli spaces, so that one can write

$$\overline{\mathcal{M}}_k = \mathcal{M}_k \cup \mathbb{R}^4 \times \mathcal{M}_{k-1} \cup S^2\mathbb{R}^4 \times \mathcal{M}_{k-2} \ldots \cup S^k\mathbb{R}^4 \quad , \tag{1}$$

where $S^i\mathbb{R}^4$ denotes the i^{th} symmetric product of points of $\mathbb{R}^4$. The topological charge density, of winding number k, in $S^l\mathbb{R}^4 \times \mathcal{M}_{k-l}$ is

$$|F_k|^2 = |F_{k-l}|^2 + \sum_{i=1}^{l} 8\pi^2\delta(x - y_i) \quad , \tag{2}$$

where $y_i \in S^i\mathbb{R}^4$ are the centers of the instanton. The Dirac delta functions are the contributions of zero size instantons, which "factorize" in a fashion similar to a dilute gas approximation. This circumstance can lead to recursion relations of the type found in [8, 9], thus simplifying instanton calculations.

In this letter we briefly report on a series of calculations which establish the formal equivalence between the coefficients of the Seiberg–Witten prepotential computed in the framework of the constrained instanton calculus and those computed in its TYM counterpart. A longer report with all the details of our computations will be presented elsewhere [10].

2 Topological Yang–Mills Theory and Instanton Moduli Spaces

The relation between supersymmetric and topological theories shows up when one observes that in the former there exists a class of position–independent Green's functions [11]. If one redefines the generators of the 4–dimensional rotation group in $\mathbb{R}^4$ in a suitably twisted fashion, the $N = 2$ SYM theory gives rise to the TYM theory. With respect to this twisted Lorentz group, SUSY charges decompose as a scalar Q, a self–dual antisymmetric tensor $Q_{\mu\nu}$ and a vector Q_μ. The field content of $N = 2$ SYM consists of a gauge field A_μ, fermions $\lambda^{\dot{A}}_\alpha$ and $\bar{\lambda}^{\dot{A}}_{\dot{\alpha}}$ ($\dot{A} = 1, 2$ and $\alpha, \dot{\alpha} = 1, 2$ are supersymmetry and spin indices respectively), and a complex scalar field ϕ, all in the adjoint of the gauge group (that we take to be $SU(2)$). Under the twist the fermionic degrees of freedom are reinterpreted in the following way: $\bar{\lambda}^{\dot{A}}_{\dot{\alpha}} \to \eta \oplus \chi_{\mu\nu}$, $\lambda^{\dot{A}}_\alpha \to \psi_\mu$, where the anticommuting fields η, $\chi_{\mu\nu}$, ψ_μ are respectively a scalar, a self–dual antisymmetric tensor and a vector.

The scalar supersymmetry of TYM plays a major rôle, in that it is still an invariance of the theory when this is formulated on a generic (differentiable) four–manifold M. The Ward identities related to Q implies that *all* the observables, including the partition function itself, are topological invariants, in the sense that they are independent of the metric on M [5]. Moreover, Q has the crucial property of being nilpotent modulo gauge transformations; this allows one to interpret it as a BRST–like charge. In fact, in order to have a strictly nilpotent BRST charge, one needs to include gauge transformations with the appropriate ghost c [12]. Defining $s = s_g + Q$, where s_g is the usual BRST operator associated to the gauge symmetry, the BRST transformations for the fields read

$$sA = \psi - Dc \quad ,$$

$$
\begin{aligned}
s\psi &= -[c,\psi] - D\phi \quad , \\
sc &= -\frac{1}{2}[c,c] + \phi \quad , \\
s\phi &= -[c,\phi] \quad ,
\end{aligned}
\tag{3}
$$

whereas the anti–fields $\chi_{\mu\nu}$, $\bar{\phi}$ and $\bar{c}$ transform under the BRST symmetry as

$$
\begin{aligned}
s\chi_{\mu\nu} &= B_{\mu\nu} \quad , \\
s\bar{\phi} &= \eta \quad , \\
s\bar{c} &= b \quad .
\end{aligned}
\tag{4}
$$

$B_{\mu\nu}$, η and b are Lagrangian multipliers which transform as $sB_{\mu\nu} = 0$, $s\eta = 0$, $sb = 0$. This algebra can be interpreted as the definition and the Bianchi identities for the curvature $\widehat{F} = F + \psi + \phi$ of the connection $\widehat{A} = A + c$ on the universal bundle $P \times \mathcal{A}/\mathcal{G}$ ($P, \mathcal{A}, \mathcal{G}$ are respectively the principal bundle over M, the space of connections and the group of gauge transformations). The exterior derivative on the manifold $M \times \mathcal{A}/\mathcal{G}$ is given by $\widehat{d} = d + s$.

It is important to distinguish two cases:

1. trivial boundary conditions ($\phi = 0$), and

2. non–trivial boundary conditions $\lim_{|x|\to\infty} \phi \equiv \mathcal{A}_{00} = v\sigma^3/2i$, $v \in \mathbb{C}$,

for the field ϕ at $|x| \to \infty$. In the first situation, the geometrical framework depicted above provides us with a very nice explicit realization of the BRST operator s as the exterior derivative on the anti–instanton moduli space $\mathcal{M}^-$ [13]. This allows us to compute correlators of s–exact operators as integrals of forms on $\partial\mathcal{M}^-$ [10]. A problem arises when ϕ has non–trivial boundary conditions, since they are not compatible with the BRST algebra in (3). This is because a non–vanishing v.e.v. for ϕ implies the existence of a central charge Z in the SUSY algebra which acts on the fields as a $U(1)$ transformation, so that $s = Q + s_g$ is not nilpotent; rather one gets

$$
\begin{aligned}
(s_g + Q)^2 A &= ZA \equiv -D\phi_Z \quad , \\
(s_g + Q)^2 \psi &= Z\psi \equiv -[\phi_Z, \psi] \quad , \\
(s_g + Q)^2 \phi_Z &= Z\phi_Z \equiv 0 \quad .
\end{aligned}
\tag{5}
$$

The scalar field ϕ_Z plays the rôle of a gauge parameter and satisfies the equations $D^2\phi_Z = 0$, $\lim_{|x|\to\infty} \phi_Z = \mathcal{A}_{00}$. The $U(1)$ symmetry has to be included in the appropriate extension of the BRST operator, through the introduction of a *global* ghost field Λ related to the central charge symmetry. We define an extended BRST operator as $s = s_g + Q - \lambda Z + \frac{\partial}{\partial\lambda}$ [14], where λ is a fermionic parameter with ghost number -1 and canonical dimension zero, such that λZ has the usual quantum numbers of a BRST operator. One checks easily that s is now nilpotent. If we define the ghost field $\Lambda \equiv \lambda\phi_Z$, with the transformation $s\Lambda = \phi_Z - [c, \Lambda]$, we can finally write the modified BRST algebra in the form

$$
\begin{aligned}
sA &= \psi - D(c + \Lambda) \quad , \\
s\psi &= -[c + \Lambda, \psi] - D\phi \quad , \\
s(c + \Lambda) &= -\frac{1}{2}[c + \Lambda, c + \Lambda] + \phi \quad , \\
s\phi &= -[c + \Lambda, \phi] \quad .
\end{aligned}
\tag{6}
$$

This algebra can be derived from (3) by just making the replacement $c \longrightarrow c+\Lambda$. Notice that $\lim_{|x|\to\infty}(c+\Lambda) = \lim_{|x|\to\infty} \Lambda \equiv \lambda\mathcal{A}_{00}$.

A TYM action can be written as a pure gauge–fixing term [12] as follows,

$$S_{\rm TYM} = 2 \int d^4x \ s{\rm Tr}\Psi \quad , \tag{7}$$

where the gauge–fixing fermion is chosen to be $\Psi = \chi^{\mu\nu} F^+_{\mu\nu} - D^\mu \bar\phi \psi_\mu + \bar c \partial^\mu A_\mu$. Explicitly

$$\begin{aligned} S_{\rm TYM} \quad = \quad & 2 \int d^4x \ {\rm Tr}\Big(B^{\mu\nu} F^+_{\mu\nu} - \chi^{\mu\nu} (D_{[\mu}\psi_{\nu]})^+ + \eta D^\mu \psi_\mu + \\ & - \bar\phi (D^2\phi - [\psi^\mu, \psi_\mu]) + b \partial^\mu A_\mu + \\ & + \chi^{\mu\nu} [c + \Lambda, F^+_{\mu\nu}] - \bar\phi [c+\Lambda, D^\mu \psi_\mu] - \bar c s(\partial^\mu A_\mu) \Big) + \\ & + \int d^4x \ \partial^\mu s {\rm Tr}(\bar\phi \psi_\mu) \quad ; \end{aligned} \tag{8}$$

the last contribution comes from integrating by parts the term ${\rm Tr}(D^\mu \bar\phi \psi_\mu)$ in the gauge–fixing fermion. The functional integration over anti–fields and Lagrangian multipliers projects onto the field subspace identified by the (zero–mode) equations

$$F^+_{\mu\nu} = 0 \quad , \tag{9}$$

$$(D_{[\mu}\psi_{\nu]})^+ = 0 \quad , \tag{10}$$

$$D^\mu \psi_\mu = 0 \quad , \tag{11}$$

$$D^2 \phi = [\psi^\mu, \psi_\mu] \quad , \tag{12}$$

where (12) must be supplemented by appropriate boundary conditions on ϕ. Once the universal connection $\widehat{A}$ is given, the components $\{F, \psi, \phi\}$ of $\widehat{F}$ are in turn determined. F is anti–self–dual, and ψ is an element of the tangent bundle $T_A \mathcal{M}^-$.

Notice that the solutions to (9), (10), (11) and (12) supplemented by their boundary conditions are exactly the field configurations which are plugged into the functional integral in the context of the constrained instanton computational method [15] in $N = 2$ SYM [2, 3, 4] as *approximate* solutions of the $N = 2$ equations of motion. As it is well–known, the action obtained by twisting the $N = 2$ SYM theory (*i.e.* the action of Witten's topological field theory [5]) differs from (8) by some extra terms which are BRST–exact and correspond to a continuous deformation of the gauge–fixing [12]. The Ward identities related to the BRST symmetry guarantee that the two actions are completely equivalent as for the computation of the observables of the theory: more precisely, *the Green's functions of s–closed operators can be computed using any one of them obtaining the same result*. This is why we used, with a slight abuse of language, the same name TYM for both actions.

In the topological theory we are dealing with, functional integration reduces to an integration over $\mathcal{M}^-$ and $T_A\mathcal{M}^-$. Both in the vanishing and in the non–vanishing v.e.v. case, the computation of the Green's functions we will consider boils down to integrating differential forms on the anti–instanton moduli space, and their non–perturbative contribution will be symbolically represented by the formal expression

$$\langle fields \rangle = \int_{\mathcal{M}^-} \left[(fields) \ e^{-S_{\rm TYM}} \right]_{zero-mode \ subspace} \quad . \tag{13}$$

When ϕ has trivial boundary conditions, $S_{\rm TYM}$ vanishes on the zero–mode subspace. For winding number $k = 1$, the top form on the (8–dimensional) anti–instanton moduli space is ${\rm Tr}\phi^2(x_1){\rm Tr}\phi^2(x_2)$, and one can compute the corresponding correlation function [10]. Since the BRST operator s acts on $\mathcal{M}^-$ as the exterior derivative, we get the chain of equations [10]

$$< {\rm Tr}\phi^2 {\rm Tr}\phi^2 >= \int_{\mathcal{M}^-} {\rm Tr}\phi^2 {\rm Tr}\phi^2 = \int_{\partial\mathcal{M}^-} {\rm Tr}\phi^2 K_c = \frac{1}{2} \quad , \tag{14}$$

where the second equality follows from

$$\mathrm{Tr}\phi^2 = sK_c \quad , \quad K_c = \mathrm{Tr}\left(csc + \frac{2}{3}ccc\right) \quad , \tag{15}$$

an expression which parallels the well–known relation

$$\mathrm{Tr}F^2 = dK_A \quad , \quad K_A = \mathrm{Tr}\left(AdA + \frac{2}{3}AAA\right) \quad . \tag{16}$$

The case in which ϕ has non–trivial boundary conditions requires a more detailed analysis. The functional measure is in fact different from 1, since now the surface contribution

$$S_{\mathrm{inst}} = 2\int d^4x\ \partial^\mu s\mathrm{Tr}(\bar{\phi}\psi_\mu) \tag{17}$$

is non–vanishing. This gives rise to non–trivial correlation functions which get contribution from topological sectors of *any* instanton number k; the reason for that can be traced back to the fact that $\exp(-S_{\mathrm{inst}})$ acts as a generating functional for differential forms on $\mathcal{M}^-$. The most interesting example is $u(v) = \langle \mathrm{Tr}\phi^2\rangle$, which parametrizes the quantum vacua of the $N = 2$ theory, thus playing a prominent rôle in the context of the Seiberg–Witten solution for the $N = 2$ Wilsonian action. In the next section we will sketch how the formal equation (13) can be put to work in order to compute $u(v)$.

3 The ADHM Construction and Instanton Computations

Before doing so, we briefly recall some basic elements of the ADHM construction [6], which provides us with a parametrization of the moduli space of self–dual connections (instantons)[1], $\mathcal{M}^+$. The dimension of $\mathcal{M}^+$ is $8k$; however the ADHM description is given in terms of a redundant set of parameters, and its reparametrization symmetries will play a major rôle in the following. We will show that it is possible to realize the BRST algebra *directly* on the instanton moduli space, without having to solve any field equation. This way the construction acquires a geometrical meaning and stands on its own.

The ADHM construction, which gives all $SU(2)$ self–dual connections, is purely algebraic and we find it more convenient to use quaternionic notations. The points, x, of the quaternionic space $\mathbb{H} \equiv \mathbb{C}^2 \equiv \mathbb{R}^4$ can be conveniently represented in the form $x = x^\mu \sigma_\mu$, with $\sigma_\mu = (i\sigma^b, \mathbb{1}_{2\times 2}), b = 1, 2, 3$. The σ^b's are the Pauli matrices, and $\mathbb{1}_{2\times 2}$ is the two–dimensional identity matrix. To construct a self–dual connection of winding number k, let us introduce a $(k + 1) \times k$ quaternionic matrix linear in x, $\Delta = a + bx$, where a has the generic form

$$a = \begin{pmatrix} w_1 & \dots & w_k \\ & & \\ & a' & \\ & & \end{pmatrix} \quad ; \tag{18}$$

a' is a $k \times k$ quaternionic matrix, and b can be cast into the so–called "canonical" form

$$b = -\begin{pmatrix} 0_{1\times k} \\ \mathbb{1}_{k\times k} \end{pmatrix} \quad . \tag{19}$$

The (anti–hermitean) gauge connection can be written in the form

$$A = U^\dagger dU \quad , \tag{20}$$

[1] In the previous section we have conformed to the standard convention in topological field theories of taking the gauge curvature to be anti–self–dual. Unfortunately the literature on instanton calculus adopts the opposite convention (self–dual), to which we will conform from now on. In the following we will use the definitions and conventions of Sec. II of [4].

where U is the solution to the equation $\Delta^\dagger U = 0$, with the constraint $U^\dagger U = \mathbb{1}_{2\times 2}$ which ensures that A is an element of the Lie algebra of the $SU(2)$ gauge group. The condition of self–duality on the field strength of (20) is imposed by restricting the matrix Δ to obey

$$\Delta^\dagger \Delta = (\Delta^\dagger \Delta)^T \quad , \tag{21}$$

where the superscript T stands for transposition of the quaternionic elements of the matrix (without transposing the quaternions themselves). With the choice (19) for b, the instanton moduli space $\mathcal{M}^+$ is described in terms of a redundant set of $8k + k(k-1)/2$ ADHM collective coordinates. A is invariant under the reparametrizations $\Delta \to Q\Delta R$, where $R \in O(k)$, and

$$Q = \begin{pmatrix} q & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & R^T & \\ 0 & & & \end{pmatrix} \quad , \tag{22}$$

$q \in SU(2)$. We are now ready to work out the ADHM expression for $\widehat{A} = A + c + \Lambda$ as an educated extension of (20). We write

$$\widehat{A} = U^\dagger (d + s + \mathcal{C}) U \quad , \tag{23}$$

where $\mathcal{C}$ is the connection associated to the reparametrizations of the ADHM construction, under which it undergoes the transformation $\mathcal{C} \to Q(\mathcal{C} + s)Q^\dagger$. Every expression should be covariant with respect to this reparametrization symmetry group; this implies that ordinary derivatives on $\mathcal{M}^+$ have to be replaced by covariant ones, and s by its covariant counterpart $s + \mathcal{C}$.

We now construct the announced realization of the BRST algebra on instanton moduli space. It will emerge as the most general set of deformations of the ADHM data Δ compatible with the constraints (21). As a bonus, we will get a recipe to compute $\mathcal{C}$. The central point is that gauging the reparametrization symmetries of the ADHM description of $\mathcal{M}^+$ is precisely what is required in order to make it possible to write BRST transformations of Δ consistent with (21). To show this, let us now start by performing an infinitesimal scalar variation (that we call s for obvious reasons) of (21). We get

$$(s\Delta)^\dagger \Delta + \Delta^\dagger s\Delta = [(s\Delta)^\dagger \Delta]^T + (\Delta^\dagger s\Delta)^T \quad , \tag{24}$$

which should be read as an equation for $s\Delta$. Its solution can be written as

$$s\Delta = \mathcal{M} - \mathcal{C}\Delta \quad , \tag{25}$$

where $\mathcal{M}$ is *defined* as the matrix which satisfies

$$\Delta^\dagger \mathcal{M} = (\Delta^\dagger \mathcal{M})^T \quad . \tag{26}$$

Notice that $\mathcal{M}$ is what in standard instanton calculations would be called the fermionic zero–mode matrix. We put $\mathcal{M}$ in a form which parallels the one for a in (18), *i.e.*

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \dots & \mu_k \\ & & \\ & \mathcal{M}' & \\ & & \end{pmatrix} \quad , \tag{27}$$

$\mathcal{M}'$ being a $k \times k$ symmetric quaternionic matrix. The most general form for $\mathcal{C}$ consistent with all symmetries is

$$\mathcal{C} = \begin{pmatrix} \mathcal{C}_{00} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \mathcal{C}' & \\ 0 & & & \end{pmatrix} \quad , \tag{28}$$

where $\mathcal{C}'$ is a real antisymmetric $k \times k$ matrix; $\mathcal{C}_{00}$ is related to the asymptotic behavior of the ghost $c+\Lambda$ at infinity: $\mathcal{C}_{00} = \lambda \mathcal{A}_{00}$. If we now plug (25) into (24), $\mathcal{C}$ is determined by the equation

$$\Delta^\dagger(\mathcal{C}+\mathcal{C}^\dagger)\Delta = \Big[\Delta^\dagger(\mathcal{C}+\mathcal{C}^\dagger)\Delta\Big]^T \quad . \tag{29}$$

$\mathcal{C}'$ is a 1–form expanded on a basis of differentials of the bosonic moduli. If one substitutes back the computed expression for $\mathcal{C}$ into the first of (25), then the $\mathcal{M}$'s become in turn 1–forms on instanton moduli space $\mathfrak{M}^+$. As a specific example, let us consider the $k=2$ case in detail (for $k=1$ the connection $\mathcal{C}'$ vanishes). The ADHM bosonic matrix Δ reads

$$\Delta = \begin{pmatrix} w_1 & w_2 \\ x_1 - x & a_1 \\ a_1 & x_2 - x \end{pmatrix} = \begin{pmatrix} w_1 & w_2 \\ a_3 & a_1 \\ a_1 & -a_3 \end{pmatrix} + b(x - x_0) \quad , \tag{30}$$

where $x_0 = (x_1+x_2)/2$, $a_3 = (x_1 - x_2)/2$. The solution to the bosonic constraint (21) is simply given by

$$a_1 = \frac{1}{4|a_3|^2} a_3(\bar{w}_2 w_1 - \bar{w}_1 w_2 + \Sigma) \quad , \tag{31}$$

where Σ is an arbitrary real parameter related to the left–over $O(2)$ reparametrization invariance (which can be exploited to put Σ to zero). Setting $H = w_1^2 + w_2^2 + 4(a_1^2 + a_3^2)$, one then finds $\mathcal{C}_{00} = \lambda\mathcal{A}_{00}$, $\mathcal{C}'_{11} = \mathcal{C}'_{22} = 0$, and

$$\mathcal{C}'_{12} = -\mathcal{C}'_{21} = \frac{1}{H}\Big[\bar{w}_1(s+\mathcal{C}_{00})w_2 - \bar{w}_2(s+\mathcal{C}_{00})w_1 + 2(\bar{a}_3 s a_1 - \bar{a}_1 s a_3)\Big] \quad . \tag{32}$$

We now consider the fermionic constraint (26). Its s–variation gives

$$(\mathcal{M}^\dagger + \Delta^\dagger\mathcal{C})\mathcal{M} - [(\mathcal{M}^\dagger + \Delta^\dagger\mathcal{C})\mathcal{M}]^T = (\Delta^\dagger s\mathcal{M})^T - \Delta^\dagger s\mathcal{M} \quad , \tag{33}$$

which should be thought of as an equation for $s\mathcal{M}$. Its most general solution can be cast into the form $s\mathcal{M} = \mathcal{A}\Delta - \mathcal{C}\mathcal{M}$, where $\mathcal{A}$ has an expression which parallels (28) and must satisfy the constraint

$$\Delta^\dagger\mathcal{A}\Delta - (\Delta^\dagger\mathcal{A}\Delta)^T = (\mathcal{M}^\dagger\mathcal{M})^T - \mathcal{M}^\dagger\mathcal{M} \quad . \tag{34}$$

We want now to clarify the relation between $\mathcal{A}$ and $\mathcal{C}$. To this end, let us perform one more s–variation of $s\Delta$ and $s\mathcal{M}$; after a little algebra we get

$$\begin{aligned} s^2\Delta &= \Big(\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C}\Big)\Delta \quad , \\ s^2\mathcal{M} &= \Big(\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C}\Big)\mathcal{M} + \Big(s\mathcal{A} + [\mathcal{C},\mathcal{A}]\Big)\Delta \quad . \end{aligned} \tag{35}$$

The nilpotency of the BRST operator s implies

$$\mathcal{A} - s\mathcal{C} - \mathcal{C}\mathcal{C} = 0 \quad , \tag{36}$$

$$s\mathcal{A} + [\mathcal{C},\mathcal{A}] = 0 \quad ; \tag{37}$$

therefore it becomes possible to consistently interpret (37) as the Bianchi identity for $\mathcal{A}$, seen as the field–strength of $\mathcal{C}$ as per (36). Finally, we note that the variation of (34) simply gives an identity.

Summarizing, we have found that consistency between the s–variation of the bosonic ADHM matrix Δ and the constraint (21) yields

$$\begin{aligned} s\Delta &= \mathcal{M} - \mathcal{C}\Delta \quad , \\ s\mathcal{M} &= \mathcal{A}\Delta - \mathcal{C}\mathcal{M} \quad , \\ s\mathcal{A} &= -[\mathcal{C},\mathcal{A}] \quad , \\ s\mathcal{C} &= \mathcal{A} - \mathcal{C}\mathcal{C} \quad , \end{aligned} \tag{38}$$

where Δ and $\mathcal{M}$ satisfy (21) and (26) respectively. The set of equations (38) is our main result as it yields an explicit realization of the BRST algebra on the instanton moduli space.

Two observations are in order. First, using (3) or (6) we can work backward and deduce the explicit expressions for ψ, ϕ, getting [10]

$$\begin{aligned} \psi &= U^\dagger \mathcal{M} f (d\Delta)^\dagger U + U^\dagger (d\Delta) f \mathcal{M}^\dagger U \quad , \\ \phi &= U^\dagger \mathcal{M} f \mathcal{M}^\dagger U + U^\dagger \mathcal{A} U \quad , \end{aligned} \tag{39}$$

which coincide with the known solutions of the equations (10), (11), (12) (in (39) we set $f \equiv (\Delta^\dagger \Delta)^{-1}$). When inserted into (17), they yield exactly the supersymmetric multi–instanton action with non–zero v.e.v. for the scalar [10] previously obtained in [3]. Second, (25) gives us the opportunity to discuss the issue of the instanton measure in our framework. Let us call $\{\widehat{\Delta}_i\}$ ($\{\widehat{\mathcal{M}}_i\}$), $i = 1, \ldots, p$ where $p = 8k$, a basis of (ADHM) coordinates on $\mathcal{M}^+$ ($T_A \mathcal{M}^+$). (25) then yields $\widehat{\mathcal{M}}_i = s\widehat{\Delta}_i + (\widehat{\mathcal{C}\Delta})_i$. Therefore, the $\widehat{\mathcal{M}}_i$'s and the $s\widehat{\Delta}_i$'s are related by $\widehat{\mathcal{M}}_i = K_{ij}(\widehat{\Delta}) s\widehat{\Delta}_j$, where K_{ij} is a (moduli–dependent) linear transformation, which is completely known once the explicit expression for $\mathcal{C}$ is plugged into $\widehat{\mathcal{M}}_i$. After projection onto the zero–mode subspace, *any polynomial in the fields becomes a well–defined differential form on* $\mathcal{M}^+$. A generic function on the zero–mode subspace can be written as

$$\begin{aligned} g(\widehat{\Delta}, \widehat{\mathcal{M}}) &= g_0(\widehat{\Delta}) + g_{i_1}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} + \frac{1}{2!} g_{i_1 i_2}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} \widehat{\mathcal{M}}_{i_2} + \ldots \\ &+ \frac{1}{p!} g_{i_1 i_2 \ldots i_p}(\widehat{\Delta}) \widehat{\mathcal{M}}_{i_1} \widehat{\mathcal{M}}_{i_2} \cdots \widehat{\mathcal{M}}_{i_p} \quad , \end{aligned} \tag{40}$$

the coefficients of the expansion being totally antisymmetric in their indices. It then follows that

$$\int_{\mathcal{M}^+} g(\widehat{\Delta}, \widehat{\mathcal{M}}) = \int_{\mathcal{M}^+} s^p \widehat{\Delta} \; |\mathrm{det} K| g_{12\ldots p}(\widehat{\Delta}) \quad , \tag{41}$$

where $s^p \widehat{\Delta} \equiv \prod_{i=1}^p s\widehat{\Delta}_i$. This formula is an operative tool to calculate physical amplitudes, and the determinant of K naturally stands out as the instanton integration measure for $N = 2$ SYM theories. In standard instanton calculations this important ingredient is obtained as a ratio of bosonic and fermionic zero–mode Jacobians. In our framework it emerges instead in a geometrical and straightforward way, *without the need of any computations of ratios of determinants nor of any knowledge of the explicit expressions of bosonic and fermionic zero–modes.* The only ingredient is the connection $\mathcal{C}$. The instructive exercise of computing K and its determinant (*i.e.* the instanton measure) in the cases of winding number equal to one and two [10] gives results which agree with previously known formulae [3].

To make clear how the strategy for computing instanton–dominated correlators works in our set–up, we now focus the attention on the Green's function $\langle \mathrm{Tr}\phi^2 \rangle$, which is relevant for the computation of the Seiberg–Witten prepotential [8, 4]. In [4] it was shown that the topological sector of winding number k contributes to it a

$$< \frac{\mathrm{Tr}\phi^2}{8\pi^2} >_k = -k \int_{\mathcal{M}^+ \setminus \{x_0\}} e^{-[S_{\mathrm{inst}}]_k} \quad , \tag{42}$$

where $\mathcal{M}^+ \setminus \{x_0\}$ is the "reduced" moduli space obtained after first integrating over the instanton center x_0; the dimension of $\mathcal{M}^+ \setminus \{x_0\}$ is $4n$, where $n = 2k - 1$. Let us call $\{\widetilde{\Delta}_i\}$, $i = 1, \ldots, n$ a set of ADHM data for $\mathcal{M}^+ \setminus \{x_0\}$, and $\widetilde{\mathcal{M}}_i$, $i = 1, \ldots, n$ its fermionic counterpart. The exponential of the fermionic part $[S_F]_k$ of $[S_{\mathrm{inst}}]_k$ can now be expanded in powers. Under the integration over the reduced moduli space $\mathcal{M}^+ \setminus \{x_0\}$, the only surviving term of the expansion will come from the top form on $\mathcal{M}^+ \setminus \{x_0\}$. It is crucial to remark that all the terms containing $\mathcal{C}_{00}$ do not contribute to the amplitudes since the parameter λ does not belong to the moduli space. After some algebra one finds

$$< \frac{\mathrm{Tr}\phi^2}{8\pi^2} >_k = -k \cdot (32\pi^2)^{2n} \int_{\mathcal{M}^+ \setminus \{x_0\}} s^{4n} \widetilde{\Delta} \;\; \det h |\det K| e^{-[S_B]_k} \quad , \tag{43}$$

where $[S_B]_k$ is the bosonic part of the instanton action, and h is defined through the formula $[S_F]_k = 8\pi^2 \overline{\widetilde{\mathcal{M}}}_i^{\dot{A}\alpha} (h_{ij})_\alpha{}^\beta (\widetilde{\mathcal{M}}_j)_{\beta\dot{A}}$ $(i, j = 1, \ldots, n)$. Explicit computations of $< \mathrm{Tr}\phi^2/8\pi^2 >_k$ for $k = 1$ and $k = 2$ from (43) give complete agreement with previously known results [2, 3, 4]. Details will be presented elsewhere [10].

A new interesting possibility now arises [10] observing that it is possible to write the action S_{inst} as the s–variation of a certain function of the moduli, more precisely in the form

$$[S_{\text{inst}}]_k = [S_B + S_F]_k = 4\pi^2 s\Big\{ \mathrm{Tr}\Big[\bar{v}(\sum_{i=1}^{k} \mu_i \bar{w}_i - w_i \bar{\mu}_i)\Big]\Big\} \quad , \tag{44}$$

with $s[S_{\text{inst}}]_k = 0$. As explained above, in order to compute (42), one has to extract from $\exp(-[S_{\text{inst}}]_k)$ the top form on $\mathcal{M}^+\backslash\{x_0\}$, which reads

$$e^{-[S_{\text{inst}}]_k}\Big|_{top\ form} = 4\pi^2 s\left\{ \mathrm{Tr}\Big[\bar{v}(\sum_{i=1}^{k} \mu_i \bar{w}_i - w_i \bar{\mu}_i)\Big]([S_F]_k)^{4k-3}([S_B]_k)^{-4k+2}\left(1 - e^{-[S_B]_k}\sum_{l=0}^{4k-3} \frac{([S_B]_k)^l}{l!}\right)\right\} . \tag{45}$$

In fact, (45) enables one to write $< \mathrm{Tr}\phi^2 >_k$ as an integral over the boundary of the instanton moduli space. The circumstance that Green's functions can be computed in principle on the boundary of $\mathcal{M}^+$ may greatly help in calculations, since instantons on $\partial\mathcal{M}^+$ obey a kind of dilute gas approximation [10], as emphasized in the Introduction. We leave to future work the computations with $k > 1$, limiting here our attention to the $k = 1$ case.

From the analyses of [7, 16], it is known that the boundary of the $k = 1$ moduli space consists of instantons of zero "conformal" size; this means that if we projectively map the Euclidean flat space $\mathbb{R}^4$ onto a 4–sphere S^4, the boundary of the corresponding transformed $k = 1$ instanton moduli space is given by instantons of zero conformal size τ, where τ we is obtained from $|w|$ through a projective transformation ($|w|$ itself does not yield a globally defined coordinate on the S^4 instanton moduli space). In terms of the size $|w|$ of the $\mathbb{R}^4$ instanton, the $\tau \to 0$ limit corresponds to $|w| \to 0, \infty$. Specializing (45) to $k = 1$ and inserting it in (42) we get

$$\begin{aligned} < \frac{\mathrm{Tr}\phi^2}{8\pi^2} >_{k=1} &= -4\pi^2 \int_{\mathcal{M}^+\backslash\{x_0\}} s\Big\{ \mathrm{Tr}\Big[\bar{v}(\mu\bar{w} - w\bar{\mu})\Big][S_F]_{k=1}\frac{1}{[S_B]^2_{k=1}} \cdot \\ &\quad \cdot\Big(1 - e^{-[S_B]_{k=1}} - [S_B]_{k=1}e^{-[S_B]_{k=1}}\Big)\Big\} \\ &= -\frac{8\pi^2}{v^2}\Big(1 - e^{-4\pi^2|v|^2|w|^2} - 4\pi^2|v|^2|w|^2 e^{-4\pi^2|v|^2|w|^2}\Big)\Big|_{|w|=0}^{|w|=\infty} \\ &= -\frac{8\pi^2}{v^2} \quad . \end{aligned} \tag{46}$$

In the second equality of (46) use of the Stokes' theorem has been made to express $\langle \mathrm{Tr}\phi^2/(8\pi^2)\rangle_{k=1}$ as an integral over the boundary of $\mathcal{M}^+\backslash\{x_0\}$, which for $k = 1$ is $\partial\mathbb{R}^+ \times S^3/\mathbb{Z}_2$. Again, (46) gives the correct answer.

We believe that this approach provides a natural and simplifying framework for studying non–perturbative effects in supersymmetric gauge theories.

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